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Backbends in Directed Percolation

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When directed percolation in a bond percolation process does not occur, any path to infinity on the open bonds will zigzag back and forth through the lattice. Backbends are the portions of the zigzags that go against the percolation direction. They are important in the physical problem of particle transport in random media in the presence of a field, as they act to limit particle flow through the medium. The critical probability for percolation along directed paths with backbends no longer than a given length n is defined as p_n. We prove that (p_n) is strictly decreasing and converges to the critical probability for undirected percolation p_c. We also investigate some variants of the basic model, such as by replacing the standard d-dimensional cubic lattice with a $(d-1)$-dimensional slab or with a Bethe lattice; and we discuss the mathematical consequences of alternative ways to formalize the physical concepts of "percolation" and "backbend."

KEY WORDS: Backbend; Bethe lattice; directed percolation; enhancement technique; particle transport; renormalization; slab.

1. INTRODUCTION

In this paper we discuss an extension of the concept of directed percolation. The physical motivation for this topic comes from the problem of particle transport in random media in the presence of a field, as studied by Ramaswamy and Barma in refs. 11 and 2, and we begin by briefly describing this process. The model chosen by Ramaswamy and Barma is as follows. For a random medium take the (unique) infinite open cluster under supercritical independent bond percolation on $(\mathbf{Z}^d, E)$, where $(\mathbf{Z}^d, E)$ denotes

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the d-dimensional cubic lattice with undirected nearest-neighbour bonds. Motion of particles under the action of a field in direction $\mathbf{e} = (1,1,...1) \in \mathbf{Z}^d$ is described by biased random walks on the random cluster, with hard-core exclusion between particles. Biased means that a step from $\mathbf{x}$ to $\mathbf{y}$ (with $\mathbf{xy} \in E$) receives greater weight when $\mathbf{y} \cdot \mathbf{e} > \mathbf{x} \cdot \mathbf{e}$.

Let us define a *path* in $\mathbf{Z}^d$ to be a finite or infinite sequence of distinct vertices $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2,..., (\mathbf{x}_k) \in \mathbf{Z}^d$ such that $\mathbf{x}_0\mathbf{x}_1, \mathbf{x}_1\mathbf{x}_2,..., (\mathbf{x}_{k-1}\mathbf{x}_k) \in E$. Note that we assume paths to be self-avoiding; we shall not exclude the possibility $k = 0$, that is, a path may have length 0. The path is said to be *directed* if

$$\mathbf{x}_0 \cdot \mathbf{e} < \mathbf{x}_1 \cdot \mathbf{e} < \mathbf{x}_2 \cdot \mathbf{e} \ldots (< \mathbf{x}_k \cdot \mathbf{e})$$

Let $d \geqslant 2$ be fixed. Write a configuration of open and closed edges in E as $\omega_1 \in \Omega_1 = \{\text{open, closed}\}^E$ and write $\mathbf{P}_p$ for the percolation measure on Ω_1 with parameter $p \in [0, 1]$. Let C (respectively C_0) be the random set of vertices $\mathbf{x} \in \mathbf{Z}^d$ for which there is an open (directed) path from $\mathbf{0}$ to $\mathbf{x}$, and let

$$p_c = p_c(d) = \sup\{p\colon \mathbf{P}_p(|C| = \infty) = 0\}$$
$$p_0 = p_0(d) = \sup\{p\colon \mathbf{P}_p(|C_0| = \infty) = 0\}$$

The requirement $p > p_c$ is essential for Ramaswamy and Barma's process because for $p < p_c$ there is a.s. no infinite cluster and therefore no concept of a flow of particles. On the other hand when $p > p_0$ we expect a.s. a large net flow of particles through the infinite directed cluster. More interesting is the régime $p_c < p < p_0$. Ramaswamy and Barma show that in this case particles tend to flow through the medium along the least tortuous infinite path, it being physically harder for particles to follow paths which have long "backbends" against the direction of the field.

Formally, for $0 \leqslant n < \infty$ we say a path $\mathbf{x}_0, \mathbf{x}_1,..., (\mathbf{x}_k)$ is an *n-path* if for every i and j with $0 \leqslant i \leqslant j\ (\leqslant k)$ we have $\mathbf{x}_j \cdot \mathbf{e} \geqslant \mathbf{x}_i \cdot \mathbf{e} - n$ (that is, "the path never retreats further than n units back from its record level," or, "there is no backbend of size greater than n"), and we let C_n be the random set of vertices $\mathbf{x} \in \mathbf{Z}^d$ for which there is an open n-path from $\mathbf{0}$ to $\mathbf{x}$. (The reader should satisfy herself that for $n = 0$ this is in accordance with the definition given above for directed percolation.) The C_n define a sequence of critical probabilities p_n in analogy with p_0 above. Since every n-path is an $(n+1)$-path, we have

$$p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots \geqslant p_c \tag{1}$$

thus the régime $(p_c, p_0]$ is divided up into sub-régimes. The thesis of Ramaswamy and Barma is that their process actually exhibits a phase transition at each of the points p_n, and the net flow through the percolation cluster at parameter p is determined by which sub-régime p belongs to.

Leaving aside its physical origins, a study of the sequence (p_n) is an interesting problem in directed percolation theory. The main goal of this paper is to prove rigorously the following intuitively appealing theorem which is implicit in Ramaswamy and Barma's physics.

Theorem 1. For (p_n) defined as above, we have

$$p_0 > p_1 > p_2 > \cdots \tag{2}$$

$$p_n \to p_c \quad \text{as} \quad n \to \infty \tag{3}$$

Theorem 1 will be proved in Sections 4 and 5 below. We dedicate Sections 2, 3 and 6 to discussing some variants of the above model.

2. EXTENSIONS

In this section we decribe some extensions and variants of the model presented above.

One way of varying the model is to change the underlying lattice. It is possible to define a version of the model on a Bethe lattice, and here, in addition to proving a result corresponding to Theorem 1, we can find exact values for the sequence of critical points—details are in Section 3.

Another interesting possibility is to replace the lattice by a $(d-1)$-dimensional slab. For integers $l < r$ let

$$S(l; r) = \{\mathbf{x} \in \mathbf{Z}^d : l \leqslant \mathbf{x} \cdot \mathbf{e} \leqslant r\}$$

and (for $r \geqslant 0$) let C_n^r be the random set of vertices $\mathbf{x} \in S(-n; r)$ for which there is an open n-path in $S(-n; r)$ from $\mathbf{0}$ to $\mathbf{x}$. These sets define critical probabilities p_n^r, that is,

$$p_n^r = p_n^r(d) = \sup\{p \colon \mathbf{P}_p(|C_n^r| = \infty) = 0\}$$

and we believe it to be the case that

$$p_n^0 = p_n^1 = p_n^2 = \cdots > p_n \tag{4}$$

for all $n \geqslant 0$. Unfortunately we have been unable to prove the strict inequality in (4) in full generality. The weaker statement below is proved in Section 6.

Proposition 2. The following hold for (p_n^r) defined as above.

(a) In dimensions 2 and 3, p_1^0 is strictly greater than p_1.

(b) For all dimensions $d \geqslant 2$ and all $n \geqslant 1$, p_n^r is constant in r.

An important consequence of (4) is that $\lim_{r \to \infty} p_n^r > p_n$: this implies that any computer simulation of the model, due to its inherent finiteness, will not be able to provide any reasonable approximation of the n-path model on the entire space.

Next, we consider an alternative way of defining percolation in our model. Note that by straightforward diagonal and stationarity arguments, $\mathbf{P}(|C_n| = \infty) > 0$ if and only if there exists almost surely an infinite open n-path in the lattice. Let $\mathscr{E}_n$ be the event that there exists an infinite open n-path that goes with the field, where we say that an infinite path $\mathbf{x}_0, \mathbf{x}_1, \ldots$ *goes with the field* if $\sup_i(\mathbf{x}_i \cdot \mathbf{e}) = \infty$. (Of course $\mathbf{P}_p(\mathscr{E}_n)$ equals zero or one by Kolmogorov's Zero-One Law.) It would perhaps be more natural from a physics point of view to define the critical probabilities (p_n) in terms of the events $\mathscr{E}_n$ rather than $(|C_n| = \infty)$. In fact, it makes no difference which events we work with, as shown by the following proposition, also proved in Section 6.

Proposition 3. For any $p \in [0, 1]$ and $n \geqslant 0$,

$$\mathbf{P}_p(\mathscr{E}_n) = 1 \Leftrightarrow \mathbf{P}_p(|C_n| = \infty) > 0$$

We turn now to an alternative formalisation of the idea of backbends. We shall say a path $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, (\mathbf{x}_k)$ is an *n-walk* if there is no i such that

$$\mathbf{x}_i \cdot \mathbf{e} > \mathbf{x}_{i+1} \cdot \mathbf{e} > \ldots > \mathbf{x}_{i+n} \cdot \mathbf{e} > \mathbf{x}_{i+n+1} \cdot \mathbf{e}$$

or equivalently, if for every i with $0 \leqslant i$ (and $i+n+1 \leqslant k$) we have $\mathbf{x}_{i+n+1} \cdot \mathbf{e} \geqslant \mathbf{x}_i \cdot \mathbf{e} - n + 1$ (in other words, "the path never makes more than n consecutive backward steps"). Note that every n-path is also a n-walk, but that the two notions are equivalent only for n equal to 0 or 1 (see Fig. 1). Our reason for introducing an alternative here is that the physics literature is not consistent on this point and Ramaswamy and Barma seem to use the two forms interchangeably. Fortunately, most of our results do remain true under the alternative formalisation. Let $\tilde{C}_n, \tilde{p}_n, \tilde{\mathscr{E}}_n$ be the n-walk versions of $C_n, p_n, \mathscr{E}_n$ defined above. In the Bethe lattice set-up, the versions of $\tilde{p}_n$ can be computed exactly, in a similar way to the (p_n), and details of this are also given in Section 3. On the cubic lattice Theorem 1 continues to hold when tildes are added: the proof of (2) is similar to that given below for the n-path, so we shall omit it; and for the limit (3) there

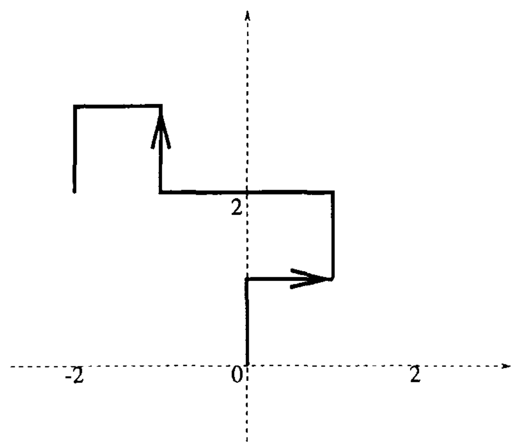

Fig. 1. This path is a 2-walk and a 3-path, but not a 2-path.

is in fact nothing further to prove, since every n-path is also an n-walk and therefore $\tilde{p}_n \leqslant p_n$ for all $n \geqslant 1$. A bigger surprise is in store for us in the case of percolation on slabs (Section 6), where there is qualitatively different behaviour in the n-walk set-up. (4) becomes

Proposition 4. Defining the critical probabilities $\tilde{p}_n^r$ in the natural way, we have

$$\tilde{p}_n^0 > \tilde{p}_n^1 > \tilde{p}_n^2 > \cdots > \tilde{p}_n$$

for all $n \geqslant 2$ and in all dimensions $d \geqslant 3$.

The following problems about n-walk percolation are still open:

1. Does $\lim_k \tilde{p}_n^k = \tilde{p}_n$?
2. Prove or disprove a version of Proposition 3 for n-walks, that is, that there exists a.s. an infinite open n-walk going with the field if and only if $\mathbf{P}_p(|\tilde{C}_n| = \infty) > 0$.

Finally, we consider the following generalization of the definition of n-path. Let $\sigma\colon \mathbf{N} \to \mathbf{N}$, and call a path $\mathbf{x}_0, \mathbf{x}_1, \ldots, (\mathbf{x}_k)$ a *σ-path* if for every i and j with $0 \leqslant i \leqslant j$ ($\leqslant k$) we have $\mathbf{x}_j \cdot \mathbf{e} \geqslant \mathbf{x}_i \cdot \mathbf{e} - \sigma(\mathbf{x}_i \cdot \mathbf{e})$. Thus if a path has reached a level n (that is, $\mathbf{x}_i \cdot \mathbf{e} = n$), it is allowed a backbend of length $\sigma(n)$. We can define the critical probability $p_{(\sigma)}$ as

$$p_{(\sigma)} = \sup\{p\colon \mathbf{P}_p(|C_\sigma| = \infty) = 0\}$$

where C_σ is the random set of vertices $\mathbf{x} \in \mathbf{Z}^d$ for which there is an open σ-path from $\mathbf{0}$ to $\mathbf{x}$. Note that for $\sigma(n) = k$ (a constant) we are in the k-backbend model, with $p_{(\sigma)} = p_k$, while if $\sigma(n) = n$ we are in the situation of undirected bond percolation in the half space and $p_{(\sigma)} = p_c$. For general σ, it is an easy consequence of Theorem 1 that $\lim_{n \to \infty} \sigma(n) = \infty$ implies $p_{(\sigma)} = p_c$. An interesting question is now: how does the percolation model behave with different functions σ? In particular, given that directed percolation and undirected percolation are believed to belong to two different universality classes, it would be interesting to investigate the dependence on σ of various critical exponents.

3. BETHE LATTICE

The notion of backbends extends very naturally to a Bethe lattice setting. Using a multi-type branching process argument, we show that the critical probabilities in this setting can be expressed simply in terms of eigenvalues of certain matrices, and so percolation-theoretical questions reduce to problems of matrix manipulation. Exact values of the critical probabilities can then by worked out with the help of the computer. This situation is in contrast to the cubic lattice model, where we must recourse to hard arguments of probability theory to prove our results, and exact values are beyond our reach.

We restrict ourselves to considering the rooted Bethe lattice with coordination number 4. Our arguments are applicable to Bethe lattices of arbitrary coordination number, but the attraction of this particular one is that it is easily represented diagrammatically (see Fig. 2) in such a way as to point the analogy between it and the $\mathbf{Z}^2$ square lattice, with all bonds lying either "North/South" or "East/West," and a field being thought of as acting in the "north-easterly" direction.

In analogy with the quantity $\mathbf{x} \cdot \mathbf{e}$ defined in the previous section, we define the *depth* $\eta(v)$ of a vertex v in the Bethe lattice recursively as follows.

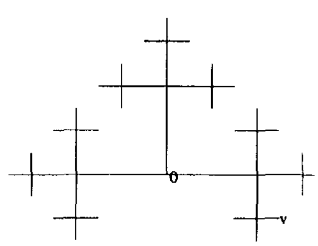

Fig. 2. Part of the rooted Bethe lattice with coordination number 4.

The root 0 has depth $\eta(0)=0$, and given a vertex v with depth $\eta(v)$, its immediate neighbours to the North and East have depth $\eta(v)+1$ and its immediate neighbours to the South and West have depth $\eta(v)-1$. Thus for v as in Fig. 2 we have $\eta(v)=1$. We now have a natural formalisation of the idea of backbends in the Bethe lattice as follows: a path $\pi=v_0, v_1,..., (v_k)$ is defined to be

- an *n-path* if $\eta(v_j)\geqslant\eta(v_i)-n$ for every i and j with $0\leqslant i\leqslant j(\leqslant k)$;

and

- an *n-walk* if there is no $i\geqslant 0$ such that $\eta(v_i)>\eta(v_{i+1})>\cdots>\eta(v_{i+n+1})$.

As is customary, we impose a probability structure on the lattice by declaring an edge open with probability p and closed with probability $1-p$ independently of all other edges; (C'_n) and $(\tilde{C}'_n)$ are the sets of vertices v such that the unique path from 0 to v is an open n-path (n-walk); and (p'_n), $(\tilde{p}'_n)$ are defined in the usual way.

Suppose we now consider $\tilde{C}'_n$ as the set of individuals of a multi-type branching process, with 0 the progenitor, and the children of an individual $u\in\tilde{C}'_n$ being those $v\in\tilde{C}'_n$ such that uv is the last edge in the unique open path from 0 to v. The progenitor 0 is of type 0, and if a parent u is of type t then its children to the South and West are of type $t+1$ and its children to the North and East of type 0. Then it is a consequence of the definition of an n-walk that no individual can have type $t>n$, and the expected number of children of type j from a parent of type i (with $i, j\leqslant n$) is given by

$$\tilde{M}_{n,p}(i,j)=\begin{cases}2p & \text{if } (i,j)=(0,0)\\ p & \text{if } (i,j)=(0,1)\\ p & \text{if } i\geqslant 1 \text{ and } j=0\\ 2p & \text{if } i\geqslant 1 \text{ and } j=i+1\\ 0 & \text{otherwise}\end{cases}$$

The event $|\tilde{C}'_n|=\infty$ now corresponds to the survival of the multi-type branching process. But by the Perron-Frobenius Theorem, the offspring matrix $\tilde{M}_{n,p}$ has a positive real eigenvalue $\tilde{\lambda}_{n,p}$ such that

$$\tilde{\lambda}_{n,p}=\max\{|\tilde{\lambda}|: \tilde{\lambda} \text{ is an eigenvalue of } \tilde{M}_{n,p}\}$$

and it is well-known (see for example ref. 10) that the process will survive with positive probability if and only if $\tilde{\lambda}_{n,p}>1$. Thus,

$$\tilde{p}'_n=(\tilde{\lambda}_{n,1})^{-1}$$

Using MATLAB we obtain the following values (to 4 decimal places from $n=1$ onwards):

$$\tilde{p}'_0=0.5, \quad \tilde{p}'_1=0.4142, \quad \tilde{p}'_2=0.3761, \quad \tilde{p}'_3=0.3576, \quad \tilde{p}'_4=0.3478$$

We can follow a similar approach for n-paths, this time saying that if a vertex $u \in C'_n$ is of type t then its children to the South and West are of type $t+1$ but those to the North and East are of type $\max\{t-1, 0\}$. This again yields labels from 0 to n for every vertex in C'_n, but now it is not so easy to write down an offspring matrix: a vertex of type 1, for example, will potentially have two children of type 2 if its parent is of type 0, but only one if its parent is itself of type 2. We get around this by thinking of the *edges* between vertices in C'_n as the individuals of our new multi-type branching process, with the type of an edge being a pair of numbers given by the types of the two vertices it joins (the parent first). Thus an edge can have either type $(0, 0)$ or type (i, j) for $0 \leqslant i,j \leqslant n$ with $|i-j|=1$, and the offspring matrix is now given by

$$M_{n,p}((i, j)(k, l)) = \delta_{jk}(1+\delta_{ij}\,\delta_{il}+(1-\delta_{ij})(1-\delta_{il}))\,p$$

(where $\delta_{ij}=\mathbf{1}\{i=j\}$). The same procedure as before yields the values:

$$p'_0=0.5, \quad p'_1=0.4142, \quad p'_2=0.3795, \quad p'_3=0.3631, \quad p'_4=0.3542$$

We remark that these numerical results are consistent with those obtained non-rigorously by Barma and Ramaswamy[2].

Finally we give a Bethe lattice version of Theorem 1.

Proposition 5. For (p'_n) defined as above, we have

$$p'_0 > p'_1 > p'_2 > \cdots p'_n \to 1/3 \qquad \text{as} \quad n \to \infty \tag{5}$$

Moreover, these statements remain true if the (p'_n) are replaced by $(\tilde{p}'_n)$.

Proof. For the same reasons as given in the $\mathbf{Z}^d$ case, we shall prove the statements only for the (p'_n): the proof of (5) for n-walks is similar to that given below, and the limit $\tilde{p}'_n \to 1/3$ is immediate since $\tilde{p}'_n \leqslant p'_n$ for all n.

Let λ_n then be the n-path equivalent of $\tilde{\lambda}_{n,1}$ above, and let $g_n = g_n(\lambda)$ be the characteristic equation of $M_{n,1}$, so λ_n is the largest real zero of g_n.

We shall show that (λ_n) is strictly increasing and that $\lambda_n \to 3$ as $n \to \infty$. Writing out g_n as a determinant we obtain after some algebraic manipulation

$$g_n = \lambda^2 g_{n-1} - (g_{n-1} + 4g_{n-2} + \cdots + 4^{n-2} g_1 + 4^{n-1})$$

for $n \geqslant 3$. This allows us to express $g_n - 4g_{n-1}$ as a telescopic sum yielding

$$g_n - (\lambda^2 + 3)\, g_{n-1} + 4\lambda^2 g_{n-2} = 0 \tag{6}$$

Since every n-path is also an $(n+1)$-path, (p'_n) is nonincreasing and thus (λ_n) is nondecreasing. Suppose now that $\lambda_N = \lambda_{N+1}$ for some N. Then (6) implies that λ_N is an eigenvalue of $M_{n,1}$ for all n. By inspection of $M_{n,1}$ for a few small values of n we see that such a common eigenvalue does not exist; hence, (λ_n) is strictly increasing.

To prove the limit $\lambda_n \to 3$ note that since the row sums of $M_{n,1}$ all equal either 1 or 3, we have $\lambda_n \in [1, 3]$ for all n. But for $\lambda \in (1, 3)$ the polynomial equation associated with (6)

$$m^2 - (\lambda^2 + 3)\, m + 4\lambda^2 = 0$$

has complex roots and so (6) has general solution of the form

$$g_n = A r^n \cos(n\theta + \alpha) \tag{7}$$

where A, r, θ and α are functions of $\lambda \in (1, 3)$. Given any $\varepsilon > 0$, it is straightforward to check that these functions are continuous and that θ is not constant on the interval $(3 - \varepsilon, 3)$; and hence, for n large enough, g_n has a zero in this interval. ∎

4. THE LIMIT

In this section we prove (3) that the critical probabilities for n-path percolation converge to that for undirected bond percolation, i.e., $\lim_{n \to \infty} p_n = p_c$. We shall give two distinct proofs for the separate cases $d \geqslant 3$ and $d = 2$. In the latter case we shall use a strictly two-dimensional argument involving box crossings. In the former case we apply a result of Grimmett and Marstrand that holds only in dimensions 3 or above.

Firstly then, suppose $d \geqslant 3$. Since every undirected path in the slab $S(-n(d+2); n(d+2))$ is a $2n(d+2)$-walk, we have

$$p_c \leqslant p_{2n(d+2)} \leqslant p_c(S(-n(d+2); n(d+2))) \tag{8}$$

where $p_c(U)$ denotes the critical probability of undirected bond percolation restricted to a set $U \subset \mathbf{Z}^d$. Let

$$F = \{(x^1, x^2, ..., x^d) \in S(0; 1) : x^i = 0 \text{ if } i \geq 4\}$$
$$B(n) = [-n, n]^d$$
$$2nF + B(n) = \{2n\mathbf{x} + \mathbf{y} : \mathbf{x} \in F, \mathbf{y} \in B(n)\}$$

and note that $S(-n(d+2); n(d+2)) \supset 2nF + B(n)$, hence,

$$p_c(S(-n(d+2); n(d+2))) \leq p_c(2nF + B(n))$$

Now from (1) and (8) it suffices to prove

$$p_c(2nF + B(n)) \to p_c \qquad \text{as} \quad n \to \infty \tag{9}$$

But, as remarked in Stacey[12], the sub-lattice generated by F is isomorphic to the two-dimensional hexagonal lattice (see Fig. 3). In particular, by Wierman[13], $p_c(F) = 1 - 2\sin(\pi/18) < 1$ and so (9) follows by the Grimmett-Marstrand Theorem for bond percolation (Theorem 7.8 of Grimmett[7]), given below as Theorem 6.

Theorem 6. If F is an infinite connected subset of $\mathbf{Z}^d$ with $p_c(F) < 1$, then $\lim_{n \to \infty} p_c(2nF + B(n)) = p_c$.

Let us now turn to the case $d = 2$.

For $0 \leq q \leq 1$, we call a probability measure on Ω_1 *1-dependent with parameter* q if it is such that each edge $\mathbf{x}_1\mathbf{x}_2 \in E$ is open with probability q and if this is independent of the status of any edge $\mathbf{x}_3\mathbf{x}_4 \in E$ whenever

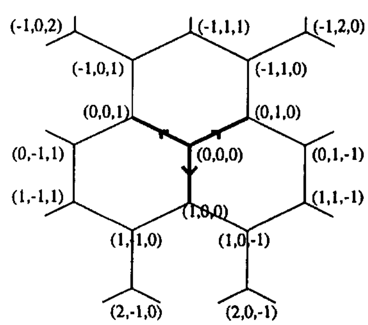

Fig. 3. Part of the set F (in the case $d = 3$) seen in projection on the plane $\mathbf{x} \cdot \mathbf{e} = 0$. The standard basis of coordinate vectors is shown in bold.

$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are all distinct. The following proposition concerning directed percolation in 1-dependent models can be proved by an elementary contour argument analogous to that in Durrett[3], §10 (or directly from a general result by Liggett *et al.*[8]).

Proposition 7. There exists $q_0 < 1$ such that $\mu(|C_0| = \infty) > 0$ for every 1-dependent measure μ with parameter $q > q_0$.

A *top-bottom crossing* of a box $[a, b] \times [c, d] \subset \mathbf{Z}^2$ is a (undirected) path in the box from $[a, b] \times \{c\}$ to $[a, b] \times \{d\}$; a *left-right crossing* is defined similarly. Let $\mathscr{A}_n$ be the event that there is an open top-bottom crossing of $[-n, n] \times [-n, 5n]$ as well as an open left-right crossing of $(0, 4n) + B(n)$ (see Fig. 4).

Fix $p > p_c$. By a standard argument we have $\mathbf{P}_p(\mathscr{A}_n) \to 1$ as $n \to \infty$ (see for example the proof of Theorem 9.23 in Grimmett[6]). Choose n with $\mathbf{P}_p(\mathscr{A}_n) > q_0$. We shall show by a renormalisation technique that this choice of n satisfies $p > p_{8n}$, and (3) then follows since our choice of $p > p_c$ is arbitrary.

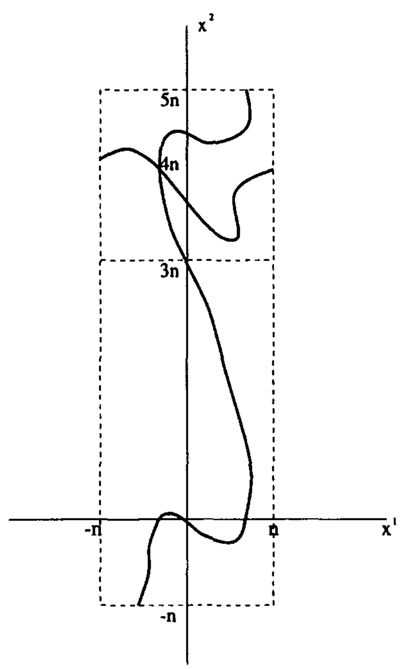

Fig. 4. An occurrence of the event $\mathscr{A}_n$.

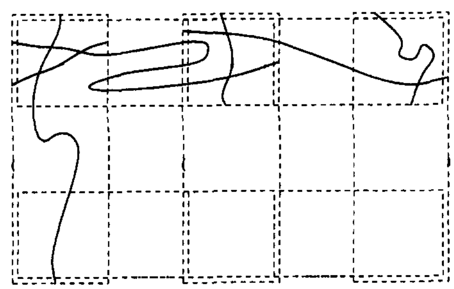

Fig. 5. Part of the grid made up of $2n \times 6n$ boxes (dotted lines): the curves (in bold) represent open paths in the lattice E. The box on the left and the two boxes along the top of the figure are "good".

As shown in Fig. 5, we form a cris-cross grid in E using infinitely many $2n \times 6n$ boxes, and say that a box is *good* if a suitable version of $\mathscr{A}_n$ occurs within it. Note that the status of two different boxes is independent if and only if the boxes do not overlap. Thus we can think of the grid G of good and bad boxes as a 1-dependent bond percolation model, with parameter $\mathbf{P}_p(\mathscr{A}_n) > q_0$. Hence $|C_0| = \infty$ in G with positive probability; as remarked previously, this is equivalent to the a.s. existence of an open infinite directed path in G.

Now suppose $\mathbf{y}_1, \mathbf{y}_2,...$ is an infinite open path in G, that is, $(\mathbf{y}_1 \mathbf{y}_2)$, $(\mathbf{y}_2 \mathbf{y}_3)$,... denotes a sequence of good boxes in the grid with $(\mathbf{y}_{i-1} \mathbf{y}_i) \cap (\mathbf{y}_i \mathbf{y}_{i+1}) \neq \varnothing$ for all $i \geqslant 2$. By suitably concatenating box crossings we can find a corresponding infinite open path $\mathbf{x}_1, \mathbf{x}_2...$ in E such that for all $i < j$ there exist $k \leqslant l$ with $\mathbf{x}_i \in (\mathbf{y}_k \mathbf{y}_{k+1})$ and $\mathbf{x}_j \in (\mathbf{y}_l \mathbf{y}_{l+1})$. Since

$$\max_{\mathbf{x}, \mathbf{x}' \in Q} (\mathbf{x} \cdot \mathbf{e} - \mathbf{x}' \cdot \mathbf{e}) = 8n$$

for any box Q in the grid, it follows that $\mathbf{x}_1, \mathbf{x}_2...$ is an $8n$-path in E if $\mathbf{y}_1, \mathbf{y}_2,...$ is a directed path in G. Thus to every infinite open directed path in G there corresponds an infinite open $8n$-path in E, and so the previous paragraph implies that C_{8n} is infinite in E with positive probability, as required.

5. STRICT MONOTONICITY

To prove (2), we begin be introducing some notation to see that backbend models can be viewed as "enhancements" in the sense of Aizenman and Grimmett[1] and Menshikov.[9] It then remains to check how general enhancement techniques can be applied in our specific model.

Label each edge in E (independently of the other edges and of the open/closed configuration) "special" with probability s and "dull" with probability $1-s$, and write $\Omega_2 = \{\text{special}, \text{dull}\}^E$. We write the enhanced configuration as $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ and denote the measure on $\Omega_1 \times \Omega_2$ by $\mathbf{P}_{p,s}$. Expectation with respect to $\mathbf{P}_{p,s}$ will be denoted by $\mathbf{E}_{p,s}$.

Given a path $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots (\mathbf{x}_k)$ we say it is an *$n^\star$-path* if it is an n-path and if the edge $\mathbf{x}_{j-1}\mathbf{x}_j$ is special whenever $\mathbf{x}_j \cdot \mathbf{e} = \mathbf{x}_i \cdot \mathbf{e} - n$ (for $0 \leqslant i < j$ $(\leqslant k)$). We write $C_n^\star$ for the random set of vertices $\mathbf{x} \in \mathbf{Z}^d$ for which there is an open $n^\star$-path from $\mathbf{0}$ to $\mathbf{x}$. Note:

- When $s=0$, almost surely no edges are special and so an $n^\star$-path is the same as an $(n-1)$-path, thus

$$\mathbf{P}_{p,0}(|C_n^\star| = \infty) = \mathbf{P}_p(|C_{n-1}| = \infty)$$

- When $s=1$, almost surely all edges are special and so every n-path is an $n^\star$-path, thus

$$\mathbf{P}_{p,1}(|C_n^\star| = \infty) = \mathbf{P}_p(|C_n| = \infty)$$

By the latter remark, it is sufficient for (2) to prove that for each finite $n \geqslant 1$ there exists $p < p_{n-1}$ such that

$$\mathbf{P}_{p,1}(|C_n^\star| = \infty) > 0 \tag{10}$$

Fix $n \geqslant 1$; for $R \geqslant n+1$ let

$$H(R) = B(R) \cap S(-dn-1; d(R-1))$$

(see Fig. 6). Given $U \subset \mathbf{Z}^d$ we define ∂U to be the set of vertices $\mathbf{x} \in U$ such that $\mathbf{xy} \in E$ for some $\mathbf{y} \notin U$; let $\mathscr{K}_R$ be the event that there is an open $n^\star$-path from $\mathbf{0}$ to some vertex of $\partial(H(R))$.

For any $R \geqslant n+1$, any edge $f \in E$, and any configuration $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, we say that f is *pivotal* (respectively *$\star$-pivotal*) for $\mathscr{K}_R$ if the configuration obtained from (ω_1, ω_2) by setting f to be open (resp. special) is in $\mathscr{K}_R$, but the configuration obtained by setting f to be closed (resp. dull) is not in $\mathscr{K}_R$. Let N_R denote the (random) number of pivotal edges for the event $\mathscr{K}_R$, and similarly let $N_R^\star$ be the number of $\star$-pivotal edges for $\mathscr{K}_R$.

The following lemma is analogous to Lemma 2 of Aizenman and Grimmett[1].

Lemma 8. There exists a strictly positive continuous function $g = g(p, s)$ on $(0, 1)^2$ such that

$$\mathbf{E}_{p,s}(N_R^\star) \geqslant g(p, s)\, \mathbf{E}_{p,s}(N_R)$$

for all $R \geqslant n+1$.

It follows by a version of Russo's formula (Lemma 1 of Aizenman and Grimmett[1]) that

$$\frac{\partial}{\partial s} \mathbf{P}_{p,s}(\mathscr{K}_R) \geqslant g(p, s) \frac{\partial}{\partial p} \mathbf{P}_{p,s}(\mathscr{K}_R)$$

and the proof of (10) is completed by elementary differential calculus (as detailed in Aizenman and Grimmett).

For the proof of Lemma 8 we shall assume that d and n are fixed with $d = 2$ and $n \geqslant 3$. Proofs for higher dimensions and/or lower values of n are similar.

As a preliminary we first need some more notation.

Given $U \subset \mathbf{Z}^2$ and $f \in E$ we shall say that f is *incident with* U if $f \cap U$ is nonempty; we define the *interior* of U by $\operatorname{int}(U) = U \backslash \partial U$. Let G be the group of $\mathbf{Z}^2$ actions generated by all translations on $\mathbf{Z}^2$ together with the map $(x^1, x^2) \mapsto (x^2, x^1)$; for $\theta \in G$ we shall write U^θ for the image of U under θ.

Let S, T be the sets

$$S = \{(x^1, x^2) \in \mathbf{Z}^2 : 0 \leqslant x^1 \leqslant 3,\ -1 \leqslant x^2 \leqslant n,\ x^1 + x^2 \leqslant n+1\}$$

$$T = \{(1, n), (2, n-1), (3, n-2)\}$$

and let $\gamma_1, \gamma_2, \gamma_3$ be paths in S from T to $(1, 0)$ given by:

$$\gamma_1 = (1, n), (1, n-1), \ldots, (1, 1), (1, 0)$$

$$\gamma_2 = (2, n-1), (2, n-2), (2, n-3), \ldots, (2, 1), (2, 0), (1, 0)$$

$$\gamma_3 = (3, n-2), (2, n-2), (2, n-3), \ldots, (2, 1), (2, 0), (1, 0)$$

(see Fig. 6).

Proof of Lemma 8. Fix $R \geqslant n+1$. Given an edge $f = \mathbf{xy}$ with $\mathbf{x}, \mathbf{y} \in H(R)$, let Π_f be the number of edges incident with $f + B(n)$ that are $\star$-pivotal for $\mathscr{K}_R$. Suppose (ω_1, ω_2) is a configuration for which f is pivotal for $\mathscr{K}_R$ but $\Pi_f = 0$. The main idea of the proof is to find a way of modifying

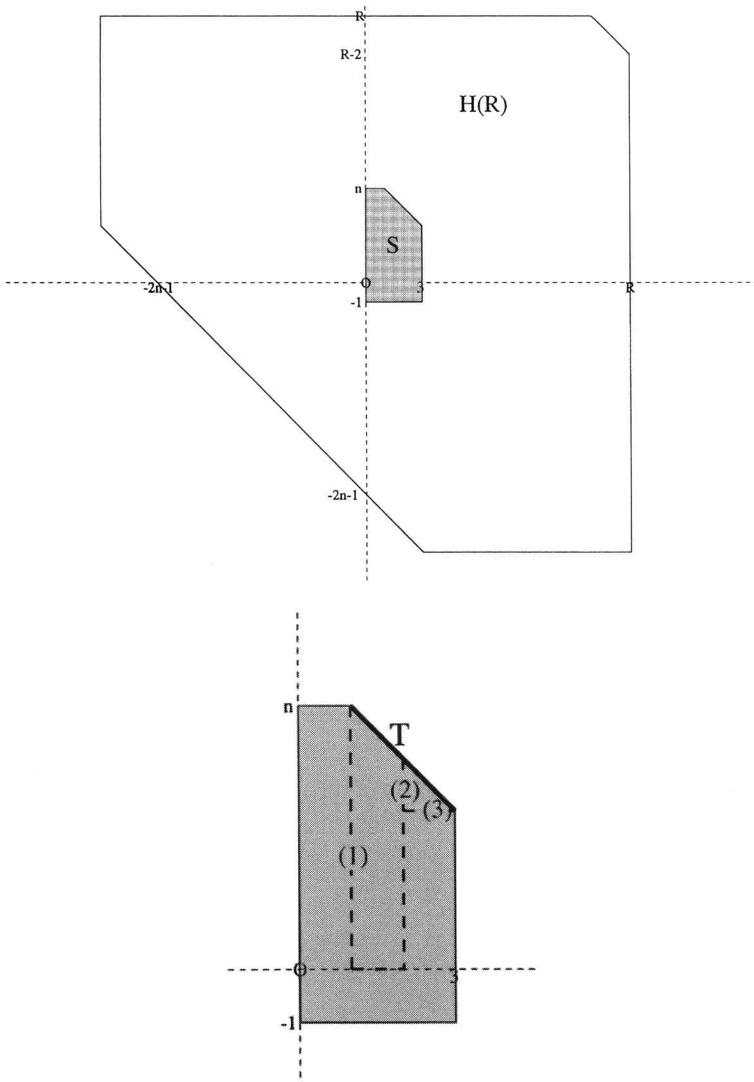

Fig. 6. $H(R)$ and S when $n = 5$, $R = 14$ and $d = 2$. In the detail (bottom left) the paths γ_1, γ_2 and γ_3 in S are marked (1), (2), (3) respectively.

this configuration by changing the status of some of the edges in $f + B(n)$, in such a way that Π_f becomes strictly positive. It will follow, as in Lemma 2 of Aizenman and Grimmett, that the mean numbers of pivotal and $\bigstar$-pivotal edges are comparable, uniformly in R.

Finding a suitable modification of (ω_1, ω_2) is a question of working through some elementary graph theory. Note that by the definition of an n-path we must have

$$\min\{\mathbf{x} \cdot \mathbf{e}, \mathbf{y} \cdot \mathbf{e}\} \geqslant -n$$

Therefore it follows from the geometry of H_R that there is some $\theta \in G$ such that $\mathbf{x}, \mathbf{y} \in S^\theta \subset H_R$, f is incident with T^θ, and $\mathbf{0} \notin \operatorname{int}(S^\theta)$. (Without loss of generality suppose $\mathbf{x} \in T^\theta$.) By choice of f and (ω_1, ω_2), f is in some n-path π from $\mathbf{0}$ to $\partial H(R)$ such that π is open in the configuration obtained from (ω_1, ω_2) by setting f itself to be open. Let $\mathbf{s}$ be the first point of π in S^θ and $\mathbf{t}$ be the last. Since $\mathbf{x}, \mathbf{y} \in S^\theta$ it follows that $\mathbf{s}$ and $\mathbf{t}$ are distinct elements of ∂S^θ. Let π_1, π_3 denote the (possibly empty) sub-paths of π from $\mathbf{0}$ to $\mathbf{s}$ and from $\mathbf{t}$ to $\partial H(R)$. Write $\pi_1 = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k$ (where $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{x}_k = \mathbf{s}$, $k \geqslant 0$).

We shall find a path $\pi_2 = \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_l$ in S^θ from $\mathbf{s}$ to $\mathbf{t}$ (where $\mathbf{x}_l = \mathbf{t}$, $k < l$) so that the concatenation ρ of π_1, π_2, π_3 is an n-path from $\mathbf{0}$ to $\partial H(R)$, and so that $\mathbf{x}_i \cdot \mathbf{e} = \mathbf{x}_j \cdot \mathbf{e} - n$ holds for some i, j with $0 \leqslant j < i$ and $k \leqslant i \leqslant l$. If we now modify the configuration (ω_1, ω_2) by making the edges of ρ incident with S^θ open and special, and all other edges incident with S^θ closed and dull, then ρ must become the only open n-path from $\mathbf{0}$ to $\partial H(R)$; and the edge $\mathbf{x}_{i-1}\mathbf{x}_i$ becomes $\bigstar$-pivotal for $\mathscr{K}_R$. This shows that we can modify the configuration (ω_1, ω_2) within the box $f + B(n)$ in such a way that Π_f become non-zero. Since the size of $f + B(n)$ is finite and independent of both R and f, it follows that there is a finite, positive, continuous function δ on $(0, 1)^2$, independent of R and of f, such that

$$\mathbf{P}_{p,s}(f \text{ is pivotal for } \mathscr{K}_R, \Pi_f = 0) \leqslant \delta(p, s)\, \mathbf{P}_{p,s}(\Pi_f \geqslant 1)$$

Thus

$$\mathbf{P}_{p,s}(f \text{ is pivotal for} \mathscr{K}_R) \leqslant (\delta(p, s) + 1)\, \mathbf{E}_{p,s}(\Pi_f)$$

Summing now over all such edges $f = \mathbf{xy}$ with $\mathbf{x}, \mathbf{y} \in H(R)$,

$$\begin{aligned} \mathbf{E}_{p,s}(N_R) &\leqslant (\delta(p, s) + 1) \sum_f \mathbf{E}_{p,s}(\Pi_f) \\ &\leqslant 8(n+2)^2\, (\delta(p, s) + 1)\, \mathbf{E}_{p,s}(N_R^\bigstar) \end{aligned}$$

and the lemma follows.

It remains to be shown how we choose the path π_2. Now there exists a path from $\mathbf{s}$ to T^θ in $\partial S^\theta \backslash \{\mathbf{t}\}$; and this can be concatenated with exactly one of the paths $\gamma_1^\theta, \gamma_2^\theta, \gamma_3^\theta$ from T^θ to $(1, 0)^\theta$. Denote the resulting path $\pi_2' = \mathbf{x}_k', \mathbf{x}_{k+1}',..., \mathbf{x}_m'$, where $\mathbf{x}_k' = \mathbf{x}_k = \mathbf{s}$, $\mathbf{x}_m' = (1, 0)^\theta$, $m > k$.

We shall call a vertex $\mathbf{x}_r'$ of π_2' *marginal* if $\mathbf{x}_r' \cdot \mathbf{e} \leqslant \mathbf{x}_i \cdot \mathbf{e} - n$ for some $i = i(r) < r$. (Note that here we are abusing notation slightly and taking $\mathbf{x}_i$ to mean $\mathbf{x}_i'$ when $i \geqslant k$).

By construction, $\mathbf{x}_{m-n}' \in T^\theta$ and $\mathbf{x}_{m-n}' \cdot \mathbf{e} - \mathbf{x}_m' \cdot \mathbf{e} = n$, so certainly $\mathbf{x}_m'$ is marginal. On the other hand, for (ω_1, ω_2) the edge $\mathbf{x}_{k-1}\mathbf{x}_k$ (if it is defined) is supposed not to be $\bigstar$-pivotal for $\mathscr{K}_R$, therefore $\mathbf{x}_k$ is not marginal.

Choose r minimal such that $\mathbf{x}_r'$ is a marginal vertex in π_2'. The above discussion shows that r is well-defined and greater than k. By minimality it is easy to see that $i = i(r)$ satisfies

$$\mathbf{x}_r' \cdot \mathbf{e} = \mathbf{x}_i \cdot \mathbf{e} - n \tag{11}$$

and that $\mathbf{x}_r' \cdot \mathbf{e} \geqslant \mathbf{x}_h \cdot \mathbf{e} - n$ for all $h \leqslant r$, thus $\mathbf{x}_0, \mathbf{x}_1,..., \mathbf{x}_{r-1}', \mathbf{x}_r'$ is an n-path.

At this point let us put an end to all abusive notation and declare $\mathbf{x}_k,..., \mathbf{x}_{r-1}$ to equal $\mathbf{x}_k',..., \mathbf{x}_{r-1}'$. Now we claim that

$$\mathbf{t} \cdot \mathbf{e} \geqslant \mathbf{x}_r' \cdot \mathbf{e} \tag{12}$$

If $i \leqslant k$ then this is clear from (11) since $\mathbf{x}_i$ and $\mathbf{t}$ both lie on the n-path π. For the case $i > k$, we note that $\mathbf{x}_i \cdot \mathbf{e} \leqslant \mathbf{x} \cdot \mathbf{e}$ and that $\mathbf{x}$ and $\mathbf{t}$ both lie on π, then apply a similar argument. Because of (12) and our construction of $\mathbf{x}_k,..., \mathbf{x}_{r-1}$, it is possible to extend this path to $\pi_2 = \mathbf{x}_k, \mathbf{x}_{k+1},..., \mathbf{x}_l$ from $\mathbf{s}$ to $\mathbf{t}$ (in S^θ) such that $\min_{r \leqslant j \leqslant l}(\mathbf{x}_j \cdot \mathbf{e}) = \mathbf{x}_r' \cdot \mathbf{e}$; and π_2 has the properties we required. (See Fig. 7 for some examples.) ∎

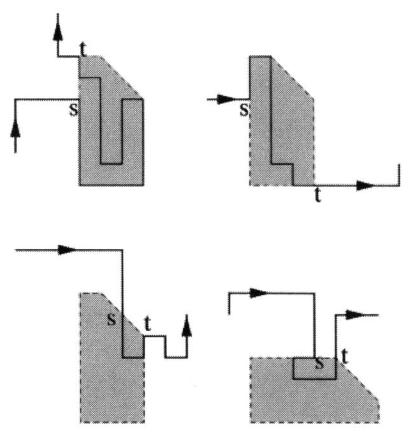

Fig. 7. Some possible configurations of ρ through the set S^θ.

6. SLABS

In this section we discuss the critical probabilities for percolation on slabs, p_n^r and $\tilde{p}_n^r$, and we show how these can be used to prove Proposition 3. For much of the section we shall use the equivalent formalisation of percolation discussed in Section 2, namely the a.s. existence of an open path in the lattice.

Proof of Proposition 2. (a) We shall in fact prove that in dimensions 2 and 3, $p_1^0 > p_0$; this is a stronger result by (1). When $d=2$ the argument is trivial (and can easily be extended to arbitrary values of n): here, the slab $S(-1; 0)$ is simply a one-dimensional line, and so $p_1^0 = p_c(1) = 1$; on the other hand it is well known (see e.g. Durrett[4]) that $p_0(2) < 1$. When $d=3$, $S(-1; 0)$ is the two-dimensional hexagonal lattice and so $p_1^0 = 1 - 2\sin(\pi/18)$ (see Section 4 above), and the result follows by the upper bound $p_0(3) < 0.473$ of Stacey[12].

(b) Fix $d \geqslant 2$ and $n \geqslant 1$. Clearly, p_n^r is monotonically decreasing in r; suppose it is not monotonically increasing. Then $p_n^0 \geqslant p_n^{r-1} > p > p_n^r$ for some $p \in [0, 1]$ and $r \geqslant 1$. It follows that there exists almost surely an open infinite n-path that is within $S(-n; r)$ but not within $S(-n; r-1)$. Any such path must include a vertex $\mathbf{z}$ with $\mathbf{z} \cdot \mathbf{e} = r$, the path from $\mathbf{z}$ onward then being contained in $S(r-n; r)$. Thus we have almost surely an infinite open n-path in $S(r-n; r)$. By stationarity it follows that $p \geqslant p_n^0$, which is a contradiction. ∎

Proposition 3 can now be proved as follows.

Proof of Proposition 3. Fix $p \in [0, 1]$ and $n \geqslant 0$. Since every infinite n-path either goes with the field or is contained in some slab $S(l; r)$, it is sufficient to prove that if there exists a.s. an open infinite n-path contained in some slab then $\mathbf{P}(\mathscr{E}_n) = 1$. So suppose such an n-path exists a.s. By (the proof of) Proposition 2(b) it follows that $S(-n; 0)$ contains an infinite n-path a.s. In particular, $S(-n; 0)$ contains an infinite cluster a.s. (i.e., the random subgraph of $(\mathbf{Z}^d, E)$ formed by all open edges $\mathbf{x}, \mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in S(-n; 0)$ has an infinite connected component almost surely); by Theorem 1′ of Gandolfi *et al.*[5], the infinite cluster is a.s. unique. By countable additivity therefore, a.s. each of the sets $S(n(i-1); ni)$ contains a unique infinite cluster K_i (for $i = 0, 1, 2, \ldots$). On this event, any given $\mathbf{z}$ with $\mathbf{z} \cdot \mathbf{e} = ni$ lies in $K_i \cap K_{i+1}$ with positive probability, so there exists a.s. some sequence $\mathbf{z}_0, \mathbf{z}_1, \ldots$ with $\mathbf{z}_i \in K_i \cap K_{i+1}$. Now $\mathbf{z}_i, \mathbf{z}_{i+1} \in K_{i+1}$ implies that there is an open path from $\mathbf{z}_i$ to $\mathbf{z}_{i+1}$ in $S(ni; n(i+1))$; by construction of $S(ni; n(i+1))$ such a path must be an n-path; and the concatenation of all these n-paths goes with the field. ∎

We conclude with a brief discussion of n-walks on slabs. The proof of Proposition 4 is similar to that of (2) in Section 5, so we shall just give a sketch here.

Fix $n \geqslant 2$ and $r \geqslant 1$, and assume $d \geqslant 3$. We want to show that $\tilde{p}_n^{r-1} > \tilde{p}_n^r$. First note that the assumption on d is made because we have $0 < \tilde{p}_n^{r-1} < 1$ if and only if $d \geqslant 3$, and these strict inequalities are necessary in order to be able to apply the usual enhancement techniques. We now introduce the customary new variable $s \in [0, 1]$, declaring all edges in $S(r-1; r)$ to be open with probability ps and all other edges open with probability p (independently), thus

$$\mathbf{P}_{p,0}(|C_n^r| = \infty) = \mathbf{P}_p(|C_n^{r-1}| = \infty)$$

and

$$\mathbf{P}_{p,1}(|C_n^r| = \infty) = \mathbf{P}_p(|C_n^r| = \infty)$$

We define a sequence of boxes $H'(R)$ with $\bigcup H'(R) = S(-n; r)$ and let $\mathscr{K}_R'$ be the event that $C_n^r \backslash H'(R)$ is nonempty. As usual the important point now is to show that if f is an edge in $S(-n; r-1)$ and ω is a configuration for which f is pivotal for $\mathscr{K}_R'$, then we can find a modification of ω of "bounded cost" for which some edge in $S(r-1; r)$ and near f becomes pivotal for $\mathscr{K}_R'$. This is done in a similar way to Section 5, by diverting an n-walk through f so that it must go to $S(r-1; r)$ and then back to its original course—making sure that on its return journey from $S(r-1; r)$ it never makes more than n backward steps consecutively, and thus remains an n-walk. Note that here we use a crucial property of n-walks for $n \geqslant 2$, that they can go arbitrarily far against the field, that is, given any $n \geqslant 2$ and $m \geqslant 1$ we can find an n-walk $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k$ with $\mathbf{x}_0 \cdot \mathbf{e} - \mathbf{x}_k \cdot \mathbf{e} = m$. This is in contrast to n-paths (and also 0-walks and 1-walks) where we must by definition have $\mathbf{x}_0 \cdot \mathbf{e} - \mathbf{x}_k \cdot \mathbf{e} \leqslant n$.

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